

RATIONAL HOMOTOPY THEORY OF MAPPING SPACES VIA LIE THEORY FOR L_∞ -ALGEBRAS

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ABSTRACT. We calculate the higher homotopy groups of the Deligne-Getzler ∞ -groupoid associated to an L_∞ -algebra and we describe Sullivan models for its connected components. As an application, we present a new approach to the rational homotopy theory of mapping spaces. For a connected space X and a nilpotent space Y of finite type, the mapping space $\text{Map}(X, Y_{\mathbb{Q}})$ is homotopy equivalent to the ∞ -groupoid associated to the L_∞ -algebra $A \hat{\otimes} L$, where A is a commutative differential graded algebra model for X and L is an L_∞ -algebra model for Y . This enables us to calculate Sullivan models for the components of $\text{Map}(X, Y_{\mathbb{Q}})$.

1. INTRODUCTION

In [17] Getzler associates an ∞ -groupoid $\gamma_\bullet(\mathfrak{g})$ to a nilpotent L_∞ -algebra \mathfrak{g} which generalizes the Deligne groupoid of a nilpotent differential graded Lie algebra [16, 17, 19, 23]. The ∞ -groupoid $\gamma_\bullet(\mathfrak{g})$ is equivalent to the ‘integral’ $\int \mathfrak{g}$ [22] and to the ‘nerve’ $\Sigma(\mathfrak{g})$ [23]. In this paper, we consider an extension of Getzler’s construction to *complete* L_∞ -algebras, i.e., inverse limits of towers of central extensions of nilpotent L_∞ -algebras. Our first main result is a calculation of the higher homotopy groups of $\gamma_\bullet(\mathfrak{g})$, which is new even for nilpotent \mathfrak{g} .

Theorem 1.1. *For a complete L_∞ -algebra \mathfrak{g} there is a natural group isomorphism*

$$B: H_n(\mathfrak{g}^\tau) \rightarrow \pi_{n+1}(\gamma_\bullet(\mathfrak{g}), \tau), \quad n \geq 0,$$

where \mathfrak{g}^τ denotes the L_∞ -algebra \mathfrak{g} twisted by the Maurer-Cartan element τ . The group structure on $H_0(\mathfrak{g}^\tau)$ is given by the Campbell-Hausdorff formula.

We stress that we put no boundedness or finiteness constraints on \mathfrak{g} . This will be essential for our applications. Another reason for pointing this out is that the non-negatively graded case is well-understood by classical rational homotopy theory: Non-negatively graded complete \mathfrak{g} of finite type correspond to Sullivan models of nilpotent spaces X of finite type, via the Chevalley-Eilenberg construction $C^*(\mathfrak{g})$. In this situation, it is known that $\gamma_\bullet(\mathfrak{g})$ is a \mathbb{Q} -localization of X , and Theorem 1.1 recovers the result that the rational homotopy groups of a nilpotent space are dual to the indecomposables of its minimal Sullivan model [34]; see also [22, §6].

As a corollary of Theorem 1.1 we obtain models for the components of $\gamma_\bullet(\mathfrak{g})$.

Corollary 1.2. *Let \mathfrak{g} be a complete L_∞ -algebra. The component of $\gamma_\bullet(\mathfrak{g})$ that contains a given Maurer-Cartan element τ is homotopy equivalent to $\gamma_\bullet(\mathfrak{g}_{\geq 0}^\tau)$, where $\mathfrak{g}_{\geq 0}^\tau$ denotes the truncation of \mathfrak{g}^τ . Hence, provided \mathfrak{g} is of finite type, the Chevalley-Eilenberg construction $C^*(\mathfrak{g}_{\geq 0}^\tau)$ is a Sullivan model for the component of $\gamma_\bullet(\mathfrak{g})$ that contains τ .*

Our main application is within the rational homotopy theory of mapping spaces. The key to this application is the following result.

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Theorem 1.3. *Let X be a connected space and let Y be a nilpotent space of finite \mathbb{Q} -type. If A is a differential graded algebra model for X and L is an L_∞ -algebra model for Y , then there is a homotopy equivalence of Kan complexes*

$$\mathrm{Map}(X, Y_{\mathbb{Q}}) \simeq \gamma_\bullet(A \hat{\otimes} L).$$

This appears as Theorem 5.2 in the text. The following result is a direct consequence of Theorem 1.1, Corollary 1.2 and Theorem 1.3.

Theorem 1.4. *Let X be a connected space and let Y be a nilpotent space of finite \mathbb{Q} -type. Let A be a commutative differential graded algebra model for X and let L be an L_∞ -algebra model for Y . There is bijection*

$$[X, Y_{\mathbb{Q}}] \cong \pi_0 \gamma_\bullet(A \hat{\otimes} L) = \mathrm{MC}(A \hat{\otimes} L) / \sim.$$

Moreover, given a map $f: X \rightarrow Y_{\mathbb{Q}}$ whose homotopy class corresponds to the equivalence class of a Maurer-Cartan element $\tau \in A \hat{\otimes} L$, the Chevalley-Eilenberg construction $C^*((A \hat{\otimes} L)_{\geq 0}^\tau)$ is a Sullivan model for the component $\mathrm{Map}(X, Y_{\mathbb{Q}})_f$ of the mapping space that contains f . In particular, there are bijections

$$H_n(A \hat{\otimes} L^\tau) \xrightarrow{\cong} \pi_{n+1}(\mathrm{Map}(X, Y_{\mathbb{Q}}), f), \quad n \geq 0.$$

For $n \geq 1$ this is an isomorphism of rational vector spaces, and for $n = 0$ it is an isomorphism of groups where the left hand side is given the Campbell-Hausdorff group structure. The Lie bracket on the left hand side corresponds to the Whitehead product on the right hand side.

Rational models for mapping spaces have been studied before by many authors, see for instance [3, 6, 8, 11, 15, 20, 27] and the recent survey [33, §3.1]. Buijs-Félix-Murillo [11] prove the existence of an L_∞ -algebra structure on the space of derivations from a Quillen model for the source to a Quillen model for the target that models a given component of the (based) mapping space, but they do not give a direct description of the L_∞ -algebra structure. What is new with our approach is that we get explicit and computable L_∞ -algebra models for the components that are expressed in terms of natural constructions on models for the source and target — tensoring, twisting by a Maurer-Cartan element and truncating. Moreover, there are no constraints such as cofibrancy or minimality on the models for the source and target. In the last section we have included a number of calculational examples.

During the preparation of this paper, we learned that variants of Theorem 1.4 have been obtained independently by Buijs-Félix-Murillo [12] and Lazarev [25] using different methods.

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2. L_∞ -ALGEBRAS

This section contains more or less standard material on L_∞ -algebras.

Graded vector spaces. We work over a ground field \mathbb{k} of characteristic zero. A graded vector space is a collection $V = \{V_i\}_{i \in \mathbb{Z}}$ of vector spaces over \mathbb{k} . We say that V is of *finite type* if each component V_i is finite dimensional. We say that it is *bounded below (above)* if $V_i = 0$ for $i \ll 0$ ($i \gg 0$), *bounded* if it is bounded below and above, and *finite* if it is bounded and of finite type. We use the convention that $V^i = V_{-i}$. We think of upper degrees as cohomological and lower degrees as homological. The graded vector space $\mathrm{Hom}(V, W)$ is defined by $\mathrm{Hom}(V, W)_i = \prod_{p+q=i} \mathrm{Hom}(V^p, W_q)$. The dual graded vector space is denoted $V^\vee = \mathrm{Hom}(V, \mathbb{k})$, where \mathbb{k} is viewed as a graded vector space concentrated in

degree 0. For an integer n , the n -fold suspension $V[n]$ of a graded vector space V is defined by $V[n]_i = V_{i-n}$.

L_∞ -algebras. An L_∞ -algebra is a \mathbb{Z} -graded vector space $\mathfrak{g} = \{\mathfrak{g}_i\}_{i \in \mathbb{Z}}$ together with maps

$$\ell_r: \mathfrak{g}^{\otimes r} \rightarrow \mathfrak{g}, \quad x_1 \otimes \dots \otimes x_r \mapsto [x_1, \dots, x_r], \quad r \geq 1,$$

of degree $r - 2$ satisfying the following axioms

- (Anti-symmetry) $[\dots, x, y, \dots] = -(-1)^{|x||y|}[\dots, y, x, \dots]$.
- (Generalized Jacobi identities) For every $n \geq 1$ and all $x_1, \dots, x_n \in \mathfrak{g}$,

$$\sum_{p=1}^n \sum_{\sigma} (-1)^\epsilon [[x_{\sigma_1}, \dots, x_{\sigma_p}], x_{\sigma_{p+1}}, \dots, x_n] = 0,$$

where the inner sum is over all permutations σ of $\{1, \dots, n\}$ such that $\sigma_1 < \dots < \sigma_p$ and $\sigma_{p+1} < \dots < \sigma_n$. The sign is given by

$$\epsilon = p + \sum_{\substack{i < j, \\ \sigma_i > \sigma_j}} (|x_i||x_j| + 1).$$

We will write $\delta(x) = [x]$. For $n = 1, 2, 3$, the generalized Jacobi identities are equivalent to

$$\delta^2(x) = 0,$$

$$\delta[x, y] = [\delta x, y] + (-1)^{|x|}[x, \delta y],$$

$$[x, [y, z]] - [[x, y], z] - (-1)^{|x||y|}[y, [x, z]] = \pm(\delta \ell_3 + \ell_3 \delta)(x \otimes y \otimes z).$$

In particular, \mathfrak{g} has an underlying chain complex, which we take to be homologically graded;

$$\dots \longrightarrow \mathfrak{g}_1 \xrightarrow{\delta_1} \mathfrak{g}_0 \xrightarrow{\delta_0} \mathfrak{g}_{-1} \longrightarrow \dots$$

The binary bracket ℓ_2 satisfies the Jacobi identity up to homotopy, the ternary bracket ℓ_3 being a contracting homotopy. For this reason an L_∞ -algebra may be thought of as a Lie algebra ‘up to homotopy’.

An L_∞ -algebra concentrated in degree 0 is the same thing as an ordinary Lie algebra, because in this case $\ell_r = 0$ for $r \neq 2$ for degree reasons, and the generalized Jacobi identities reduce to the classical Jacobi identity. An L_∞ -algebra \mathfrak{g} is called *abelian* if $\ell_r = 0$ for $r \geq 2$. An abelian L_∞ -algebra is the same thing as a chain complex (\mathfrak{g}, δ) . An L_∞ -algebra is called *minimal* if its differential $\delta = \ell_1$ is zero. An L_∞ -algebra with $\ell_r = 0$ for $r \geq 3$ is the same thing as a differential graded Lie algebra (dgl) $(\mathfrak{g}, \delta, [\cdot, \cdot])$.

The bar construction of an L_∞ -algebra. There is an alternative and more compact definition of L_∞ -algebras. An L_∞ -algebra structure on a graded vector space \mathfrak{g} is the same thing as a coderivation differential d on the symmetric coalgebra $\Lambda_c \mathfrak{g}[1]$. The relation between d and the brackets is given by the following: write $d = d_1 + d_2 + \dots$, where d_r is the component of d that decreases word-length by $r - 1$. Each d_r is a coderivation and is therefore determined by its restriction $d_r: \Lambda_c^r \mathfrak{g}[1] \rightarrow \mathfrak{g}[1]$. The formula

$$d_r(sx_1 \dots sx_r) = \pm s[x_1, \dots, x_r],$$

defines d in terms of the higher brackets and vice versa. The condition $d^2 = 0$ is equivalent to the generalized Jacobi identities for \mathfrak{g} .

The *bar construction* of an L_∞ -algebra \mathfrak{g} is defined to be the cocommutative differential graded coalgebra

$$\mathcal{C}_*(\mathfrak{g}) = (\Lambda_c \mathfrak{g}[1], d)$$

If \mathfrak{g} is a dg Lie algebra, then $\mathcal{C}_*(\mathfrak{g})$ coincides with the classical construction due to Quillen [31, §B.6]. The *Chevalley-Eilenberg construction* $C^*(\mathfrak{g})$ is by definition the dual commutative differential graded algebra $C^*(\mathfrak{g}) = \mathcal{C}_*(\mathfrak{g})^\vee$. If \mathfrak{g} is of finite type, then $C^*(\mathfrak{g})$ may be identified with $(\Lambda V, d)$ where ΛV denotes the free graded commutative algebra on $V = \mathfrak{g}[1]^\vee$.

Complete L_∞ -algebras.

Definition 2.1. A *complete L_∞ -algebra* is an L_∞ -algebra \mathfrak{g} together with a decreasing filtration $\mathfrak{g} = F_1\mathfrak{g} \supseteq F_2\mathfrak{g} \supseteq \dots$, such that

- (1) $[F_r\mathfrak{g}, \mathfrak{g}, \dots, \mathfrak{g}] \subseteq F_{r+1}\mathfrak{g}$.
- (2) For every r there is an N such that $[\mathfrak{g}^{\wedge \ell}] \subseteq F_r\mathfrak{g}$ for all $\ell > N$.
- (3) \mathfrak{g} is complete with respect to the topology defined by the filtration, i.e., the canonical map $\mathfrak{g} \rightarrow \varprojlim \mathfrak{g}/F_r\mathfrak{g}$ is an isomorphism.

We say that \mathfrak{g} is *profinite* if, additionally, each quotient $\mathfrak{g}/F_r\mathfrak{g}$ is finite.

Note that \mathfrak{g} is nilpotent in the sense of [17, Definition 4.2] if and only if \mathfrak{g} admits a filtration as in Definition 2.1 with $F_N\mathfrak{g} = 0$ for some N . In particular, if \mathfrak{g} is complete then each quotient $\mathfrak{g}/F_r\mathfrak{g}$ is nilpotent, and there is a sequence of central extensions of nilpotent L_∞ -algebras

$$0 \rightarrow F_r\mathfrak{g}/F_{r+1}\mathfrak{g} \rightarrow \mathfrak{g}/F_{r+1}\mathfrak{g} \rightarrow \mathfrak{g}/F_r\mathfrak{g} \rightarrow 0.$$

A comment about (2) is in order. Nilpotence for L_∞ -algebras involves two conditions — not only should bracket expressions of large enough depth vanish, but also brackets of high enough arity. Unlike in the case of groups or Lie algebras, it is not automatic that nilpotence of the quotient in a central extension implies nilpotence of the middle term. For this reason we need to impose (2). The stronger condition $[F_{i_1}\mathfrak{g}, \dots, F_{i_k}\mathfrak{g}] \subseteq F_{i_1+\dots+i_k}\mathfrak{g}$ implies both (1) and (2).

Complete L_∞ -algebras and Sullivan algebras. Recall [14, §12] that a *Sullivan algebra* is a commutative differential graded algebra of the form $(\Lambda V, d)$ where V is concentrated in positive cohomological degrees and admits a filtration

$$0 = V(-1) \subseteq V(0) \subseteq V(1) \subseteq \dots \subseteq V, \quad V = \bigcup_{k \geq 0} V(k)$$

such that $dV(k) \subseteq \Lambda V(k-1)$ for all k .

Proposition 2.2. *The Chevalley-Eilenberg construction $C^*(L)$ is a finite type Sullivan algebra if and only if L is a non-negatively graded, finite type, profinite L_∞ -algebra. Conversely, any finite type Sullivan algebra is isomorphic to $C^*(L)$ for some L .*

Proof. We may identify $C^*(L) = (\Lambda V, d)$ where $V = L[1]^\vee$. Then V is concentrated in positive cohomological degrees if and only if L is non-negatively graded. Suppose given a filtration $V(r)$ of V that exhibits $(\Lambda V, d)$ as a Sullivan algebra. Since V is of finite type, we may assume that each $V(k)$ is finite. If necessary, this may be achieved by setting $W(k) = V(0)^{\leq k} + V(1)^{\leq k-2} + V(2)^{\leq k-4} + \dots$. If we let $F_r L \subseteq L = V[1]^\vee$ be the annihilator of $V(r)$ we get a decreasing filtration

$$L = F_{-1}L \supseteq F_0L \supseteq \dots$$

Each quotient $L/F_r L$ is finite since $V(r)$ is finite. The condition $dV(r+1) \subseteq \Lambda V(r)$ translates into the condition $[F_r L, L, \dots, L] \subseteq F_{r+1}L$. Since $L/F_r L$ is finite we have $(L/F_r L)_{>p} = 0$ for some p . Since it is non-negatively graded this implies that brackets of arity $> p+2$ vanish in $L/F_r L$. \square

Maurer-Cartan elements. If \mathfrak{g} is a complete L_∞ -algebra the following series converges for every $\tau \in \mathfrak{g}_{-1}$

$$\mathcal{F}(\tau) = \sum_{k \geq 1} \frac{1}{k!} [\tau^{\wedge k}].$$

Here we write $[\tau^{\wedge k}]$ for $[\tau, \dots, \tau]$ (k copies of τ). If $\mathcal{F}(\tau) = 0$ then τ is called a *Maurer-Cartan element*, and we let $\text{MC}(\mathfrak{g})$ denote the set of all such elements. Given $\tau \in \text{MC}(\mathfrak{g})$ the series

$$[\alpha_1, \dots, \alpha_r]_\tau = \sum_{k \geq 0} \frac{1}{k!} [\tau^{\wedge k}, \alpha_1, \dots, \alpha_r], \quad r \geq 1,$$

converge and define a new L_∞ -algebra structure on the underlying graded vector space of \mathfrak{g} [17, Proposition 4.4]. Denote this L_∞ -algebra by \mathfrak{g}^τ .

Extension of scalars. Given a commutative differential graded algebra (cdga) A and an L_∞ -algebra \mathfrak{g} , we can extend scalars and form a new L_∞ -algebra $A \otimes \mathfrak{g}$ with differential and brackets defined by

$$\delta(x \otimes \alpha) = d_A(x) \otimes \alpha + (-1)^{|x|} x \otimes d_L(\alpha),$$

$$[x_1 \otimes \alpha_1, \dots, x_r \otimes \alpha_r] = (-1)^{\sum_{i < j} |\alpha_i| |x_j|} x_1 \dots x_r \otimes [\alpha_1, \dots, \alpha_r], \quad r \geq 2.$$

If \mathfrak{g} is a complete L_∞ -algebra and A is a cdga, we define a new complete L_∞ -algebra

$$A \hat{\otimes} \mathfrak{g} := \varprojlim (A \otimes \mathfrak{g} / F_r \mathfrak{g}).$$

This is the completion of the tensor product $A \otimes \mathfrak{g}$ with respect to the topology defined by the filtration $F_r(A \otimes \mathfrak{g}) = A \otimes F_r \mathfrak{g}$. Note that we have the relation $A \hat{\otimes} (B \hat{\otimes} \mathfrak{g}) \cong (A \otimes B) \hat{\otimes} \mathfrak{g}$.

Lemma 2.3. *If L is a profinite L_∞ -algebra, then for any cdga A there is a natural bijection*

$$\text{Hom}_{cdga}(C^*(L), A) \cong \text{MC}(A \hat{\otimes} L)$$

Proof. For finite L , this follows as in [17, Proposition 1.1]. For profinite L , the Sullivan algebra $C^*(L)$ is the union of the finitely generated Sullivan algebras $C^*(L/F_r L)$ and therefore

$$\begin{aligned} \text{Hom}_{cdga}(C^*(L), A) &\cong \varprojlim \text{Hom}_{cdga}(C^*(L/F_r L), A) \\ &\cong \varprojlim \text{MC}(A \otimes L/F_r L) = \text{MC}(A \hat{\otimes} L). \end{aligned}$$

□

Convolution L_∞ -algebras. Let C be cocommutative differential graded coalgebra (cdgc) and let L be an L_∞ -algebra. The *convolution L_∞ -algebra* $\text{Hom}(C, L)$ has differential and brackets defined by

$$\delta(f) = d_L \circ f - (-1)^{|f|} f \circ d_C,$$

$$[f_1, \dots, f_r] = \ell_n \circ (f_1 \otimes \dots \otimes f_r) \circ \Delta^{(r)}, \quad r \geq 2,$$

where $\ell_r: L^{\otimes r} \rightarrow L$ denotes the r^{th} higher bracket in L and $\Delta^{(r)}$ the iterated coproduct in C . The latter is defined inductively by $\Delta^{(2)} = \Delta: C \rightarrow C \otimes C$ and $\Delta^{(r+1)} = (\Delta^{(r)} \otimes 1) \circ \Delta: C \rightarrow C^{\otimes r+1}$.

If L is complete with respect to a filtration $F_r L$, then $\text{Hom}(C, L)$ is complete with respect to the induced filtration $F_r \text{Hom}(C, L) = \text{Hom}(C, F_r L)$. Similarly, if C is conilpotent, meaning $C = \cup C(r)$ where $C(r) = \ker(\overline{\Delta}^{(r)})$, and $\overline{\Delta}$ denotes the reduced diagonal $\overline{\Delta}(x) = \Delta(x) - x \otimes 1 - 1 \otimes x$, then $\text{Hom}(C, L)$ is complete with

respect to the filtration $F_r \operatorname{Hom}(C, L) = \operatorname{Hom}(C(r), L)$. If we let $A = C^\vee$ be the dual cdga, there is a natural morphism of complete L_∞ -algebras

$$A \hat{\otimes} L \rightarrow \operatorname{Hom}(C, L)$$

which is an isomorphism if C is of finite type and L is profinite.

3. GETZLER'S ∞ -GROUPOID $\gamma_\bullet(\mathfrak{g})$

In this section we will review the definition and some fundamental properties of the ∞ -groupoid $\gamma_\bullet(\mathfrak{g})$ associated to a complete L_∞ -algebra \mathfrak{g} .

The nerve of an L_∞ -algebra. Let Ω_\bullet denote the simplicial de Rham algebra; it is the simplicial cdga with n -simplices

$$\Omega_n = \frac{\mathbb{k}[t_0, \dots, t_n] \otimes \Lambda(dt_0, \dots, dt_n)}{(\sum_i t_i - 1, \sum_i dt_i)}, \quad |t_i| = 0, |dt_i| = 1.$$

Note that although Ω_n is not of finite type, it is bounded; $\Omega_n^i = 0$ if $i < 0$ or $i > n$.

Definition 3.1. The *nerve* of a complete L_∞ -algebra \mathfrak{g} is the simplicial set

$$\operatorname{MC}_\bullet(\mathfrak{g}) = \operatorname{MC}(\Omega_\bullet \hat{\otimes} \mathfrak{g}).$$

Since MC commutes with inverse limits we have that

$$\operatorname{MC}_\bullet(\mathfrak{g}) = \varprojlim \operatorname{MC}(\Omega_\bullet \otimes \mathfrak{g}/F_r \mathfrak{g}).$$

In particular, for nilpotent \mathfrak{g} this definition agrees with [17, 23].

Proposition 3.2 (cf. [17, Proposition 1.1]). *Let L be a finite type non-negatively graded profinite L_∞ -algebra with associated finite type Sullivan algebra $C^*(L)$. The nerve of L is isomorphic to the spatial realization of $C^*(L)$; there is an isomorphism of simplicial sets $\operatorname{MC}_\bullet(L) \cong \langle C^*(L) \rangle$.*

Proof. By definition, $\langle C^*(L) \rangle = \operatorname{Hom}_{cdga}(C^*(L), \Omega_\bullet)$, so the claim follows from Lemma 2.3. \square

Getzler's ∞ -groupoid $\gamma_\bullet(\mathfrak{g})$. For each n , the *elementary differential forms*

$$\omega_{i_0, \dots, i_k} = k! \sum_{j=0}^k (-1)^j t_{i_j} dt_{i_0} \dots \widehat{dt_{i_j}} \dots dt_{i_k} \in \Omega_n^k, \quad 0 \leq i_0 \leq \dots \leq i_k \leq n,$$

span a subcomplex $C_n \subseteq \Omega_n$. The cochain complex C_n is isomorphic to the normalized cochains $N^*(\Delta[n])$ on the standard n -simplex. The subcomplexes C_n assemble into an inclusion of simplicial cochain complexes $C_\bullet \subseteq \Omega_\bullet$. Moreover, there is a projection $P_\bullet: \Omega_\bullet \rightarrow C_\bullet$, and a contraction $s_\bullet: \Omega_\bullet^* \rightarrow \Omega_\bullet^{*-1}$ such that

$$1 - P_\bullet = ds_\bullet + s_\bullet d,$$

see [17]. Getzler introduces the simplicial subset

$$\gamma_\bullet(\mathfrak{g}) = \{\alpha \in \operatorname{MC}_\bullet(\mathfrak{g}) \mid s_\bullet \alpha = 0\} \subseteq \operatorname{MC}_\bullet(\mathfrak{g}).$$

The definition of $\gamma_\bullet(\mathfrak{g})$ extends to complete \mathfrak{g} by setting

$$\gamma_\bullet(\mathfrak{g}) = \varprojlim \gamma_\bullet(\mathfrak{g}/F_r \mathfrak{g}).$$

It follows from [17, Corollary 5.11] that the inclusion $\gamma_\bullet(\mathfrak{g}) \subseteq \operatorname{MC}_\bullet(\mathfrak{g})$ is a homotopy equivalence of Kan complexes. We now will list some properties of $\gamma_\bullet(\mathfrak{g})$ that will be useful to us.

Proposition 3.3 ([17, Proposition 5.1]). *For abelian L_∞ -algebras \mathfrak{g} there is a natural isomorphism*

$$\gamma_\bullet(\mathfrak{g}) \cong \Gamma_\bullet(\mathfrak{g}[1])$$

where $\Gamma_\bullet(\mathfrak{g}[1])$ denotes the Dold-Kan construction on the chain complex $\mathfrak{g}[1]$.

Proposition 3.4 ([17, Theorem 5.10]). *A short exact sequence of complete L_∞ -algebras*

$$0 \longrightarrow \mathfrak{g}' \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}'' \longrightarrow 0$$

induces a Kan fibration of simplicial sets

$$\gamma_\bullet(\mathfrak{g}') \longrightarrow \gamma_\bullet(\mathfrak{g}) \longrightarrow \gamma_\bullet(\mathfrak{g}'')$$

where $\gamma_\bullet(\mathfrak{g}')$ is identified with the fiber over $0 \in \text{MC}(\mathfrak{g}'')$.

4. THE HOMOTOPY GROUPS OF $\gamma_\bullet(\mathfrak{g})$

This section contains the proof of Theorem 1.1 and is the core of the paper. For \mathfrak{g} a complete L_∞ -algebra, we will define the natural map

$$B_n: H_n(\mathfrak{g}^\tau) \rightarrow \pi_{n+1}(\gamma_\bullet(\mathfrak{g}), \tau), \quad n \geq 0,$$

and prove that it is an isomorphism. We will first deal with the case $\tau = 0$. Then in Proposition 4.9 we will show how to reduce everything to this case.

To simplify notation, let $\pi_n(\mathfrak{g}) = \pi_n(\gamma_\bullet(\mathfrak{g}), 0)$ for $n \geq 1$ and $\pi_0(\mathfrak{g}) = \pi_0(\gamma_\bullet(\mathfrak{g}))$. Let

$$\omega_{i_0, \dots, i_k}^n = k! \sum_{j=0}^k (-1)^j t_{i_j} dt_{i_0} \dots \widehat{dt_{i_j}} \dots dt_{i_k} \in \Omega_n^k.$$

To simplify notation we will write

$$\begin{aligned} \omega^n &= \omega_{0 \dots n}^n = n! dt_1 \dots dt_n, \\ \omega_{\hat{r}}^n &= \omega_{0 \dots \hat{r} \dots n}^n. \end{aligned}$$

When clear from context, we will simplify notation further by dropping the superscript.

We note that the simplicial faces of $\omega_{\hat{r}}$ are given by $\partial_i(\omega_{\hat{r}}) = \omega$ if $i = r$ and zero otherwise. Furthermore, with respect to the de Rham differential, we have that $d\omega_{\hat{r}} = (-1)^r \omega$. As another notational device, let us write

$$\partial\gamma = (\partial_0\gamma, \dots, \partial_n\gamma)$$

for the simplicial boundary of an n -simplex γ .

Definition 4.1. Let \mathfrak{g} be a complete L_∞ -algebra. Define

$$B_n: H_n(\mathfrak{g}) \rightarrow \pi_{n+1}(\mathfrak{g}), \quad n \geq 0,$$

by $B_n[\alpha] = [\alpha \otimes \omega^{n+1}]$.

Proposition 4.2. *The map $B_n: H_n(\mathfrak{g}) \rightarrow \pi_{n+1}(\mathfrak{g})$ is well-defined for all $n \geq 0$.*

Proof. First of all, $\alpha \otimes \omega$ is a Maurer-Cartan element: Since $\delta\alpha = 0$ and $d\omega = 0$ we have $(d + \delta)(\alpha \otimes \omega) = 0$. Since $\omega\omega = 0$ we have $[(\alpha \otimes \omega)^{\wedge \ell}] = 0$ for all $\ell \geq 2$.

Suppose that $[\alpha] = [\beta] \in H_n(\mathfrak{g})$, say $\delta\chi = \alpha - \beta$ where $\chi \in \mathfrak{g}_{n+1}$. Then consider the degree -1 element

$$\begin{aligned} \lambda_0 &:= (d + \delta)(\chi \otimes \omega_2 - (-1)^n \alpha \otimes \omega_{\hat{0}2} - (-1)^n \beta \omega_{\hat{1}2})) \\ &= \alpha \otimes \omega_{\hat{0}} + \beta \otimes \omega_{\hat{1}} - (-1)^n \chi \otimes \omega. \end{aligned}$$

The simplicial boundary of λ_0 is

$$(1) \quad \partial\lambda_0 = (\alpha \otimes \omega, \beta \otimes \omega, 0, \dots, 0).$$

It is obvious that $(d + \delta)\lambda_0 = 0$ and that $[\lambda_0^{\wedge \ell}] = 0$ for all $\ell \geq 3$. If $n > 0$ or one of α or β is zero, then also $[\lambda_0, \lambda_0] = 0$, so that $\lambda_0 \in \gamma_{n+2}(\mathfrak{g})$ in these cases. Then (1) shows that $[\alpha \otimes \omega] = [\beta \otimes \omega] \in \pi_{n+1}(\mathfrak{g})$. In the remaining case $n = 0$ and $\alpha, \beta \neq 0$, we have that

$$(2) \quad [\lambda_0, \lambda_0] = -[\alpha, \beta] \otimes t_2 \omega_{012},$$

so λ_0 is not necessarily a Maurer-Cartan element in this case. However, using [17, Lemma 5.3] we can find an element $\lambda \in \gamma_2(\mathfrak{g})$ with the property that $\epsilon^2\lambda = 0$, $P_2R^2\lambda = \lambda_0$ and

$$(3) \quad \partial\lambda = (\alpha \otimes \omega, \beta \otimes \omega, \partial_2\lambda).$$

We will argue that $[\partial_2\lambda] = 0 \in \pi_1(\mathfrak{g})$. Then (3) shows that $[\alpha \otimes \omega] = [\beta \otimes \omega] \in \pi_1(\mathfrak{g})$.

The element λ is obtained as a limit $\lambda = \lim_{k \rightarrow \infty} \lambda_k$ where the elements λ_k are defined iteratively by

$$(4) \quad \lambda_{k+1} = \lambda_0 - \sum_{\ell \geq 2} \frac{1}{\ell!} F[\lambda_k^{\wedge \ell}], \quad k \geq 0$$

where $F = P_2h_2^2 + s_2: \Omega_2^* \rightarrow \Omega_2^{*-1}$.

Observe that since $[\alpha, \beta] = \delta[\chi, \beta]$, (2) shows that $[\lambda_0, \lambda_0]$ is a δ -boundary. Moreover, by definition $\lambda_0 \in (Z_0(\mathfrak{g}) \otimes \Omega_2^1) \oplus (\mathfrak{g}_1 \otimes \Omega_2^2)$. Assume by induction that

- $[\lambda_k, \lambda_k]$ and $\lambda_k - \lambda_0$ are δ -boundaries.
- $\lambda_k \in (Z_0(\mathfrak{g}) \otimes \Omega_2^1) \oplus (\mathfrak{g}_1 \otimes \Omega_2^2)$.

Then it follows that the same is true for λ_{k+1} . Indeed, for degree reasons $[\lambda_k^{\wedge \ell}] = 0$ for $\ell \geq 3$, so the iterative formula (4) reduces to

$$\lambda_{k+1} = \lambda_0 - \frac{1}{2} F[\lambda_k, \lambda_k].$$

This implies that $\lambda_{k+1} \in (Z_0(\mathfrak{g}) \otimes \Omega_2^1) \oplus (\mathfrak{g}_1 \otimes \Omega_2^2)$ and that $\lambda_{k+1} - \lambda_0$ is a δ -boundary. The identity

$$[\lambda_{k+1}, \lambda_{k+1}] = [\lambda_0, \lambda_0] + [\lambda_{k+1} - \lambda_0, \lambda_{k+1} + \lambda_0]$$

together with the facts that $\lambda_{k+1} - \lambda_0$ and $[\lambda_0, \lambda_0]$ are δ -boundaries and that $\lambda_k + \lambda_0 \in (Z_0(\mathfrak{g}) \otimes \Omega_2^1) \oplus (\mathfrak{g}_1 \otimes \Omega_2^2)$ imply that $[\lambda_{k+1}, \lambda_{k+1}]$ is a δ -boundary. This finishes the inductive step.

It follows that $\lambda - \lambda_0$ is a δ -boundary. Since $\partial_2\lambda_0 = 0$ this implies that $\partial_2\lambda$ is a δ -boundary. But since $\partial_2\lambda \in \gamma_1(\mathfrak{g})$ and $\partial_2\lambda \in Z_0(\mathfrak{g}) \otimes \Omega_1^1$ this implies that $\partial_2\lambda = \xi \otimes \omega_{01}$ for some δ -boundary $\xi \in \mathfrak{g}_0$. It follows from the first part of the proof that $[\partial_2\lambda] = 0 \in \pi_1(\mathfrak{g})$. \square

Proposition 4.3. *The map $B_n: H_n(\mathfrak{g}) \rightarrow \pi_{n+1}(\mathfrak{g})$ is a homomorphism of abelian groups for $n \geq 1$.*

Proof. Let $\alpha, \beta \in \mathfrak{g}_n$ be two cycles. Then we would like to show that

$$[(\alpha + \beta) \otimes \omega] = [\alpha \otimes \omega] + [\beta \otimes \omega] \in \pi_{n+1}(\mathfrak{g}).$$

This means that we have to find an element $\lambda \in \gamma_{n+2}(\mathfrak{g})$ with simplicial boundary

$$\partial\lambda = (\alpha \otimes \omega, (\alpha + \beta) \otimes \omega, \beta \otimes \omega, 0, \dots, 0).$$

We claim that

$$\lambda = \alpha \otimes \omega_{\hat{0}} + (\alpha + \beta) \otimes \omega_{\hat{1}} + \beta \otimes \omega_{\hat{2}}$$

satisfies the requirements. Indeed, λ has the correct simplicial boundary. One calculates that $(d + \delta)\lambda = 0$, and since $n \geq 1$ we have that $[\lambda^{\wedge \ell}] = 0$ for $\ell \geq 2$ for degree reasons. Therefore $\lambda \in \gamma_{n+2}(\mathfrak{g})$. \square

For a complete L_∞ -algebra \mathfrak{g} , the zeroth homology $H_0(\mathfrak{g})$ is an ordinary (ungraded) Lie algebra, and it can be given a group structure via the Campbell-Hausdorff formula

$$\alpha \cdot \beta = \log(e^\alpha e^\beta).$$

The first few terms of this formula are given by

$$\alpha \cdot \beta = \alpha + \beta + \frac{1}{2}[\alpha, \beta] + \frac{1}{12}[\alpha, [\alpha, \beta]] + \frac{1}{12}[[\alpha, \beta], \beta] + \dots$$

see e.g., [32]. The series converges as \mathfrak{g} is complete.

Proposition 4.4. *The map $B_0: H_0(\mathfrak{g}) \rightarrow \pi_1(\mathfrak{g})$ is a group homomorphism when $H_0(\mathfrak{g})$ is given the Campbell-Hausdorff group structure.*

Proof. Given cycles $\alpha, \beta \in \mathfrak{g}_0$, the product of $[\alpha \otimes \omega]$ and $[\beta \otimes \omega]$ in $\pi_1(\mathfrak{g})$ is represented by $\partial_1 \lambda$ where $\lambda \in \gamma_2(\mathfrak{g})$ has simplicial boundary

$$(5) \quad \partial \lambda = (\alpha \otimes \omega, \partial_1 \lambda, \beta \otimes \omega).$$

As in the proof of Proposition 4.3, consider the element

$$\begin{aligned} \lambda_0 &= (d + \delta)(\alpha \otimes \omega_2 - \beta \otimes \omega_0) \\ &= \alpha \otimes \omega_{12} + (\alpha + \beta) \otimes \omega_{02} + \beta \otimes \omega_{01}. \end{aligned}$$

It is not necessarily a Maurer-Cartan element, but by [17, Lemma 5.3], there is a unique element $\lambda \in \gamma_2(\mathfrak{g})$ such that $(d + \delta)h^1 \lambda = \lambda_0$. For degree reasons, the recursive formula [17, (5-20)] defining $\lambda = \lim_{k \rightarrow \infty} \lambda_k$ simplifies to

$$\lambda_{k+1} = \lambda_0 - \frac{1}{2}(P_2 h_2^1 + s_2)[\lambda_k, \lambda_k].$$

It follows that $\lambda \in \mathfrak{g}_0 \otimes \Omega_2^1$, whence $\partial_1 \lambda \in \mathfrak{g}_0 \otimes \Omega_1^1$. Since we also have $\partial_1 \lambda \in \gamma_1(\mathfrak{g})$, this element must be of the form

$$\partial_1 \lambda = \xi \otimes \omega_{01}$$

for some $\xi \in \mathfrak{g}_0$. The coefficient is determined by $\xi = I_{01}(\partial_1 \lambda) = I_{02}(\lambda)$. One finds that $I_{02}(\lambda)$ is given by the Campbell-Hausdorff formula for α and β , cf. [17, p. 296]. \square

Fibration sequences. Let $0 \rightarrow \mathfrak{g}' \xrightarrow{\mu} \mathfrak{g} \xrightarrow{\epsilon} \mathfrak{g}'' \rightarrow 0$ be a short exact sequence of complete L_∞ -algebras. Then there is an associated fibration sequence

$$\gamma_\bullet(\mathfrak{g}') \rightarrow \gamma_\bullet(\mathfrak{g}) \rightarrow \gamma_\bullet(\mathfrak{g}''),$$

whence a long exact sequence of homotopy groups

$$\cdots \rightarrow \pi_{n+2}(\mathfrak{g}'') \xrightarrow{\partial^\pi} \pi_{n+1}(\mathfrak{g}') \rightarrow \pi_{n+1}(\mathfrak{g}) \rightarrow \pi_{n+1}(\mathfrak{g}'') \rightarrow \cdots.$$

On the other hand, the short exact sequence also induces a long exact sequence in homology

$$\cdots \rightarrow H_{n+1}(\mathfrak{g}'') \xrightarrow{\partial^H} H_n(\mathfrak{g}') \rightarrow H_n(\mathfrak{g}) \rightarrow H_n(\mathfrak{g}'') \rightarrow \cdots.$$

Proposition 4.5. *Let $0 \rightarrow \mathfrak{g}' \xrightarrow{\mu} \mathfrak{g} \xrightarrow{\epsilon} \mathfrak{g}'' \rightarrow 0$ be a short exact sequence of complete L_∞ -algebras. The diagram*

$$(6) \quad \begin{array}{ccc} H_n(\mathfrak{g}'') & \xrightarrow{\partial^H} & H_{n-1}(\mathfrak{g}') \\ \downarrow (-1)^n B_n & & \downarrow B_{n-1} \\ \pi_{n+1}(\mathfrak{g}'') & \xrightarrow{\partial^\pi} & \pi_n(\mathfrak{g}') \end{array}$$

commutes for all $n \geq 1$. If \mathfrak{g}' is abelian, then it commutes for $n = 0$ also.

Proof. For a cycle $\alpha'' \in \mathfrak{g}''_n$ the class $\partial^H[\alpha'']$ is represented by any cycle $\beta' \in \mathfrak{g}'_{n-1}$ such that

$$\begin{array}{ccc} \alpha & \xrightarrow{\epsilon_n} & \alpha'' \\ \downarrow \delta & & \downarrow \delta \\ \beta' & \xrightarrow{\mu_{n-1}} & \beta \xrightarrow{\epsilon_{n-1}} 0 \end{array}$$

If we chase $[\alpha''] \in H_n(\mathfrak{g}'')$ around the diagram (6), we get the following picture:

$$\begin{array}{ccc} [\alpha''] & \xrightarrow{\partial^H} & [\beta'] \\ \downarrow (-1)^n B_n & & \downarrow B_{n-1} \\ (-1)^n [\alpha'' \otimes \omega^{n+1}] & \xrightarrow{\frac{?}{\partial^\pi}} & [\beta' \otimes \omega^n] \end{array}$$

To compute $\partial^\pi(-1)^n[\alpha'' \otimes \omega^{n+1}]$ we need to find a lift λ in the following diagram

$$\begin{array}{ccc} \Lambda^0[n+1] & \xrightarrow{(-,0,\dots,0)} & \gamma_\bullet(\mathfrak{g}) \\ \downarrow & \nearrow \lambda & \downarrow \\ \Delta[n+1] & \xrightarrow{(-1)^n \alpha'' \otimes \omega^{n+1}} & \gamma_\bullet(\mathfrak{g}'') \end{array}$$

Then $\partial_0 \lambda$ is an element in $\gamma_n(\mathfrak{g}')$ that represents $\partial^\pi(-1)^n[\alpha'' \otimes \omega^{n+1}]$, cf. [18, Ch. I.7]. We claim that

$$\lambda := (d + \delta)(\alpha \otimes \omega_{\widehat{0}}) = \beta \otimes \omega_{\widehat{0}} + (-1)^n \alpha \otimes \omega^{n+1}$$

can be chosen as a lift. Indeed, λ is a Maurer-Cartan element because $(d + \delta)\lambda = 0$ and $[\lambda^\wedge \ell] = 0$ for $\ell \geq 2$ and $n \geq 1$ for degree reasons. Furthermore, we clearly have $\epsilon(\lambda) = (-1)^n \alpha'' \otimes \omega^{n+1}$ and $\partial_i \lambda = 0$ for $0 < i \leq n+1$. It follows that $\partial^\pi(-1)^n[\alpha'' \otimes \omega^{n+1}]$ is represented by $\partial_0 \lambda = \beta \otimes \omega^n$. Thus, $\partial^\pi B_n = (-1)^n B_n \partial^H$. \square

Towers of fibrations. Let $\mathfrak{g} = \varprojlim \mathfrak{g}^{(r)}$ be an inverse limit of a tower of surjections of complete L_∞ -algebras

$$\dots \twoheadrightarrow \mathfrak{g}^{(r)} \xrightarrow{p} \mathfrak{g}^{(r-1)} \twoheadrightarrow \dots \twoheadrightarrow \mathfrak{g}^{(-1)} = 0,$$

Then we get a tower of Kan fibrations after applying the functor $\gamma_\bullet(-)$, whence a \varprojlim^1 -sequence of homotopy groups (cf. [5, Theorem IX.3.1])

$$* \longrightarrow \varprojlim^1 \pi_{n+1}(\mathfrak{g}^{(r)}) \longrightarrow \pi_n(\varprojlim \gamma_\bullet(\mathfrak{g}^{(r)})) \longrightarrow \varprojlim \pi_n(\mathfrak{g}^{(r)}) \longrightarrow *$$

There is also a \varprojlim^1 -sequence associated to the tower of surjections of chain complexes

$$0 \longrightarrow \varprojlim^1 H_{n+1}(\mathfrak{g}^{(r)}) \longrightarrow H_n(\mathfrak{g}) \longrightarrow \varprojlim H_n(\mathfrak{g}^{(r)}) \longrightarrow 0$$

Let us recall the definition of \varprojlim^1 for groups, cf. [5, IX.§2]. Given a tower of groups

$$\dots \rightarrow G_r \xrightarrow{p} G_{r-1} \rightarrow \dots \rightarrow G_{-1} = *,$$

$\varprojlim^1 G_r$ is defined as the set of equivalence classes

$$\varprojlim^1 G_r = \prod_r G_r / \sim$$

where $(x_r)_r \sim (y_r)_r$ if there is a sequence $(g_r)_r$ such that

$$y_r = g_r x_r (p g_{r+1})^{-1}, \quad \text{for all } r \geq 0.$$

Proposition 4.6. *Let $\{\mathfrak{g}^{(r)}\}_r$ be a tower of surjections of complete L_∞ -algebras and let $\mathfrak{g} = \varprojlim \mathfrak{g}^{(r)}$. The diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \varprojlim^1 H_{n+1}(\mathfrak{g}^{(r)}) & \longrightarrow & H_n(\mathfrak{g}) & \xrightarrow{f^H} & \varprojlim H_n(\mathfrak{g}^{(r)}) \longrightarrow 0 \\ & & \downarrow (-1)^k \varprojlim^1 B_{n+1} & & \downarrow B_n & & \downarrow \varprojlim B_n \\ 0 & \longrightarrow & \varprojlim^1 \pi_{n+2}(\mathfrak{g}^{(r)}) & \longrightarrow & \pi_{n+1}(\varprojlim \gamma_\bullet(\mathfrak{g}^{(r)}), 0) & \xrightarrow{f^\pi} & \varprojlim \pi_{n+1}(\mathfrak{g}^{(r)}) \longrightarrow 0 \end{array}$$

is commutative for all $n \geq 0$.

Proof. The right square commutes by naturality of the map B_n . To show that the left square commutes, we must first recall how the kernels of the maps f^H and f^π are identified with the respective \varprojlim^1 -groups.

Let $[(\alpha_r)_r]$ be an element in the kernel of $f^H: H_k(\mathfrak{g}) \rightarrow \varprojlim H_k(\mathfrak{g}^{(r)})$. This means that each $\alpha_r \in \mathfrak{g}_n^{(r)}$ is a boundary; say $\delta\beta_r = \alpha_r$. Since $p\alpha_{r+1} = \alpha_r$ for all r , the element $p\beta_{r+1} - \beta_r$ is a cycle. The identification

$$\ker f^H \xrightarrow{\cong} \varprojlim^1 H_{n+1}(\mathfrak{g}^{(r)})$$

is effected by sending the class $[(\alpha_r)_r]$ to the equivalence class represented by the sequence $([p\beta_{r+1} - \beta_r])_r \in \prod_r H_{n+1}(\mathfrak{g}^{(r)})$.

The identification

$$\ker f^\pi \xrightarrow{\cong} \varprojlim^1 \pi_{n+2}(\mathfrak{g}^{(r)})$$

goes as follows (cf. [5, IX. §3]). Given an element $[(a_r)_r]$ in the kernel of f^π , each $[a_r] \in \pi_{n+2}(\mathfrak{g}^{(r)})$ is trivial, so there are $(n+2)$ -simplices $b_r \in \gamma_{n+2}(\mathfrak{g}^{(r)})$ with simplicial boundary

$$\partial b_r = (a_r, 0, \dots, 0).$$

Choose a filler $c_r \in \gamma_{n+3}(\mathfrak{g}^{(r)})$ for the horn

$$(pb_{r+1}, b_r, -, 0, \dots, 0): \Lambda^2[n+3] \rightarrow \gamma_\bullet(\mathfrak{g}).$$

Then $\partial(\partial_2 c_r) = (0, \dots, 0)$, so $\partial_2 c_r$ represents a homotopy class in $\pi_{n+2}(\mathfrak{g}^{(r)})$. The element $[(a_r)_r]$ is sent to the equivalence class in $\varprojlim^1 \pi_{n+2}(\mathfrak{g}^{(r)})$ represented by the sequence

$$([\partial_2 c_r])_r \in \prod_r \pi_{n+2}(\mathfrak{g}^{(r)}).$$

Let $[(\alpha_r)_r]$ be an element in the kernel of f^H . The map $B_n: H_n(\mathfrak{g}) \rightarrow \pi_{n+1}(\mathfrak{g})$ sends $[(\alpha_r)_r]$ to the class $[(a_r)_r]$ where $a_r := \alpha_r \otimes \omega^{n+1}$. By commutativity of the right square this class belongs to the kernel of f^π . To prove commutativity of the left square, it suffices to prove that a filler c_r may be chosen such that

$$(7) \quad [\partial_2 c_r] = (-1)^n [(p\beta_{r+1} - \beta_r) \otimes \omega^{n+2}] \in \pi_{n+2}(\mathfrak{g}^{(r)}), \quad \text{for all } r \geq 0.$$

To this end, first observe that, in the notation above, we may choose

$$\begin{aligned} b_r &= (d + \delta)(\beta_r \otimes \omega_0) \\ &= \alpha_r \otimes \omega_0 + (-1)^{n+1} \beta_r \otimes \omega^{n+2}. \end{aligned}$$

Indeed, $\partial b_r = (a_r, 0, \dots, 0)$ and $b_r \in \gamma_{n+2}(\mathfrak{g}^{(r)})$. Next, consider the following element of $(\mathfrak{g}^{(r)} \otimes \Omega_{n+3})_{-1}$:

$$\begin{aligned} \lambda_0 &= (d + \delta)((-1)^n \alpha_r \otimes \omega_{3\dots n+3} - \beta_r \otimes \omega_{03\dots n+3} - p\beta_{r+1} \otimes \omega_{13\dots n+3}) \\ &= \alpha_r \otimes \omega_{2\dots n+3} + (-1)^{n+1} p\beta_{r+1} \otimes \omega_{1\dots n+3} \\ &\quad + (-1)^{n+1} \beta_r \otimes \omega_{02\dots n+3} + (-1)^n (p\beta_{r+1} - \beta_r) \otimes \omega_{013\dots n+3}. \end{aligned}$$

It has simplicial boundary

$$\partial\lambda_0 = (pb_{r+1}, b_r, (-1)^n(p\beta_{r+1} - \beta_n) \otimes \omega^{n+2}, 0, \dots, 0).$$

Clearly, $(d + \delta)\lambda_0 = 0$. If $n > 1$, then for degree reasons $[\lambda_0^{\wedge \ell}] = 0$ for any $\ell \geq 2$. This is also true for $k = 1$ by direct calculation. Thus, if $n > 0$ then λ_0 is a Maurer-Cartan element and we may choose $c_r = \lambda_0 \in \gamma_{n+3}(\mathfrak{g}^{(r)})$ as our filler. In this case we are done because $\partial_2\lambda_0 = (-1)^n(p\beta_{r+1} - \beta_r) \otimes \omega^{n+2}$, so that (7) is fulfilled already before passing to homotopy classes.

The case $k = 0$ requires a little more care. Then we have that

$$\lambda_0 = \alpha_r \otimes \omega_{23} - p\beta_{r+1} \otimes \omega_{123} - \beta_r \otimes \omega_{023} + (p\beta_{r+1} - \beta_r) \otimes \omega_{013}.$$

One verifies easily that $[\lambda_0^{\wedge \ell}] = 0$ for $\ell \geq 3$, but

$$\frac{1}{2}[\lambda_0, \lambda_0] = -[\alpha_r, p\beta_{r+1} - \beta_r] \otimes \frac{1}{3}t_3\omega_{0123},$$

so λ_0 is not necessarily a Maurer-Cartan element. However, by [17, Lemma 5.3] there is a unique element $\lambda \in \gamma_3(\mathfrak{g}^{(r)})$ such that $\epsilon_3^2\lambda = 0$ and $P_3R_3^2\lambda = \lambda_0$. Moreover, this element has the property that $\partial_i(\lambda) = \partial_i(\lambda_0)$ for $i \neq 2$. Therefore, we may choose $c_r := \lambda$ as a filler, and it remains to verify that (7) holds.

The element is obtained as a limit $\lambda = \lim_{k \rightarrow \infty} \lambda_k$, where λ_k is defined by the iterative formula

$$\lambda_k = \lambda_0 - \sum_{\ell \geq 2} \frac{1}{\ell!} F[\lambda_{k-1}^{\wedge \ell}], \quad k \geq 1,$$

for a certain operator¹ $F: \Omega_3^* \rightarrow \Omega_3^{*-1}$. In the first iteration, a calculation yields

$$\lambda_1 = \lambda_0 - [\alpha_r, p\beta_{r+1} - \beta_r] \otimes F\left(\frac{1}{3}t_3\omega_{0123}\right).$$

Observe that since $\alpha_r = \delta\beta_r$ and $p\beta_{r+1} - \beta_r$ is a cycle, we have that

$$[\alpha_r, p\beta_{r+1} - \beta_r] = \delta[\beta_r, p\beta_{r+1} - \beta_r],$$

so $\lambda_0 - \lambda_1$ is a δ -boundary. Moreover, $\lambda_0 - \lambda_1 \in \mathfrak{g}_1^{(r)} \otimes \Omega_3^2$. By an induction that we leave to the reader, the same is true for all k :

- $\lambda_0 - \lambda_k$ is a δ -boundary.
- $\lambda_0 - \lambda_k \in \mathfrak{g}_1^{(r)} \otimes \Omega_3^2$.

This implies that $\partial_2(\lambda_0) - \partial_2(\lambda)$ is a δ -boundary in $\mathfrak{g}_1^{(r)} \otimes \Omega_2^2$. Now, $\partial_2(\lambda_0) = (p\beta_{r+1} - \beta_r) \otimes \omega^2 \in \mathfrak{g}_1^{(r)} \otimes \Omega_2^2$. Therefore $\partial_2(\lambda) \in \mathfrak{g}_1^{(n)} \otimes \Omega_2^2$. Since also $\partial_2(\lambda) \in \gamma_2(\mathfrak{g}^{(r)})$ this implies that $\partial_2(\lambda) = \xi \otimes \omega^2$ for some cycle $\xi \in \mathfrak{g}_1^{(r)}$. Moreover, ξ and $p\beta_{r+1} - \beta_r$ differ by a boundary, whence

$$[\partial_2(\lambda)] = [(p\beta_{r+1} - \beta_r) \otimes \omega^2] \in \pi_2(\mathfrak{g}^{(r)}),$$

by well-definedness of the map $B_1: H_1(\mathfrak{g}^{(r)}) \rightarrow \pi_2(\mathfrak{g}^{(r)})$. Thus (7) is satisfied for $c_r = \lambda$, and this finishes the proof. \square

Theorem 4.7. *Let \mathfrak{g} be a complete L_∞ -algebra. The map*

$$B_n: H_n(\mathfrak{g}) \rightarrow \pi_{n+1}(\gamma_\bullet(\mathfrak{g}), 0)$$

is an isomorphism of groups for all $n \geq 0$, where $H_0(\mathfrak{g})$ is given the Campbell-Hausdorff group structure.

¹ $F = P_3h_3^2 + s_3$ in the notation of [17].

Proof. For abelian \mathfrak{g} this follows from the fact that $\gamma_\bullet(\mathfrak{g})$ is isomorphic to the Dold-Kan construction on the suspended chain complex $\mathfrak{g}[1]$ [17, Proposition 5.1]. For nilpotent \mathfrak{g} the claim follows by repeated application of Proposition 4.5 and the five lemma. Finally, for complete \mathfrak{g} it follows from Proposition 4.6 and the five lemma. \square

The components of $\gamma_\bullet(\mathfrak{g})$. So far, we have only been concerned with the homotopy groups of $\gamma_\bullet(\mathfrak{g})$ at the base-point 0. The arguments given above can be adapted to work for an arbitrary base-point $\tau \in \text{MC}(\mathfrak{g})$. Alternatively, the following will reduce everything to the base-point 0. Recall the definition of the twisted L_∞ -algebra \mathfrak{g}^τ from §2.

Lemma 4.8. *Let \mathfrak{g} be a complete L_∞ -algebra and let $\tau \in \text{MC}(\mathfrak{g})$. Then*

$$\text{MC}(\mathfrak{g}^\tau) = \{\sigma \in \mathfrak{g}_{-1} \mid \sigma + \tau \in \text{MC}(\mathfrak{g})\}.$$

Proof. After writing them out, the conditions $\sigma \in \text{MC}(\mathfrak{g}^\tau)$ and $\sigma + \tau \in \text{MC}(\mathfrak{g})$ both turn out to be equivalent to

$$\sum_{n,k \geq 0} \frac{1}{n!k!} [\sigma^{\wedge n}, \tau^{\wedge k}] = 0$$

(where an empty bracket is interpreted as zero). \square

If X is a Kan complex and v is a vertex of X , then let X_v denote the simplicial subset of X consisting of the simplices all of whose vertices are v . In other words, an n -simplex x belongs to X_v if and only if $\partial_1^{n-i} \partial_0^i x = v$ for all $0 \leq i \leq n$. The simplicial set X_v is a Kan complex and it is *reduced* in the sense that it has only one vertex. The inclusion $X_v \subseteq X$ induces a homotopy equivalence between X_v and the connected component of X containing v .

Proposition 4.9. *Let \mathfrak{g} be a complete L_∞ -algebra. For every Maurer-Cartan element τ in \mathfrak{g} there is an isomorphism of reduced Kan complexes*

$$\gamma_\bullet(\mathfrak{g})_\tau \cong \gamma_\bullet(\mathfrak{g}^\tau)_0.$$

Proof. Since, evidently, $\mathfrak{g}^\tau \hat{\otimes} \Omega_n = (\mathfrak{g} \hat{\otimes} \Omega_n)^\tau$ it follows from Lemma 4.8 that n -simplices x of $\gamma_\bullet(\mathfrak{g}^\tau)_0$ correspond bijectively to n -simplices $\tau + x$ of $\gamma_\bullet(\mathfrak{g})_\tau$, and it is obvious that this bijection respects face and degeneracy maps. \square

Theorem 4.10. *Let \mathfrak{g} be a complete L_∞ -algebra. For any Maurer-Cartan element $\tau \in \text{MC}(\mathfrak{g})$ and every $n \geq 0$ the map*

$$B_n^\tau: H_n(\mathfrak{g}^\tau) \rightarrow \pi_{n+1}(\gamma_\bullet(\mathfrak{g}), \tau), \quad B_n^\tau[\alpha] = [\tau + \alpha \otimes \omega^{n+1}],$$

is an isomorphism of groups, where $H_0(\mathfrak{g}^\tau)$ is given the Campbell-Hausdorff group structure.

Proof. This follows by combining Theorem 4.7 and Proposition 4.9. \square

For each integer m the *truncation* $\mathfrak{g}_{\geq m}$ of \mathfrak{g} is the sub L_∞ -algebra whose underlying chain complex is

$$\cdots \rightarrow \mathfrak{g}_{m+2} \xrightarrow{\delta_{m+2}^+} \mathfrak{g}_{m+1} \xrightarrow{\delta_{m+1}^+} \ker(\delta_m) \rightarrow 0 \rightarrow \cdots$$

The inclusion $\mathfrak{g}_{\geq m} \rightarrow \mathfrak{g}$ induces an isomorphism in homology $H_n(\mathfrak{g}_{\geq m}) \xrightarrow{\cong} H_n(\mathfrak{g})$ for $n \geq m$.

Corollary 4.11. *Let \mathfrak{g} be a complete L_∞ -algebra. For any Maurer-Cartan element $\tau \in \text{MC}(\mathfrak{g})$, the inclusion $\mathfrak{g}_{\geq 0}^\tau \subseteq \mathfrak{g}^\tau$ induces a homotopy equivalence of reduced Kan complexes*

$$\gamma_\bullet(\mathfrak{g}_{\geq 0}^\tau) \xrightarrow{\cong} \gamma_\bullet(\mathfrak{g})_\tau.$$

In particular, if \mathfrak{g} is of finite type, the Chevalley-Eilenberg construction $C^*(\mathfrak{g}_{\geq 0}^\tau)$ is a Sullivan model for the connected component of $\gamma_\bullet(\mathfrak{g})$ containing τ .

Proof. By Proposition 4.9 we may assume that $\tau = 0$ without loss of generality. It is clear that the image of $\gamma_\bullet(\mathfrak{g}_{\geq 0}) \rightarrow \gamma_\bullet(\mathfrak{g})$ is contained in the simplicial subset $\gamma_\bullet(\mathfrak{g})_0$. We need to show that the resulting map

$$\gamma_\bullet(\mathfrak{g}_{\geq 0}) \rightarrow \gamma_\bullet(\mathfrak{g})_0$$

is a homotopy equivalence. Since both the source and target are reduced Kan complexes, this happens if and only if the map induces an isomorphism on π_n (at the unique base-point) for all $n \geq 1$. Since $\mathfrak{g}_{\geq 0} \rightarrow \mathfrak{g}$ induces an isomorphism in homology in non-negative degrees, Theorem 4.7 together with naturality of the map B_n finishes the proof; for all $n \geq 0$

$$\begin{array}{ccc} H_n(\mathfrak{g}_{\geq 0}) & \xrightarrow[B_n]{\cong} & \pi_{n+1}(\mathfrak{g}_{\geq 0}) \\ \downarrow \cong & & \downarrow \\ H_n(\mathfrak{g}) & \xrightarrow[B_n]{\cong} & \pi_{n+1}(\mathfrak{g}). \end{array}$$

The last statement follows from Proposition 3.2. \square

Naturality. Let us be more explicit about the naturality of the map B_n^τ in Theorem 4.10. Given an L_∞ -morphism $f: \mathfrak{g} \rightarrow \mathfrak{h}$ between complete L_∞ -algebras there is an induced map $f_*: \text{MC}(\mathfrak{g}) \rightarrow \text{MC}(\mathfrak{h})$ given by

$$f_*(\tau) = \sum_{n \geq 0} \frac{1}{n!} f_n(\tau^{\wedge n}).$$

Moreover, for each $\tau \in \text{MC}(\mathfrak{g})$ there is an induced L_∞ -morphism $f^\tau: \mathfrak{g}^\tau \rightarrow \mathfrak{h}^{f_*(\tau)}$ given by

$$f_n^\tau(x_1, \dots, x_n) = \sum_{\ell \geq 0} \frac{1}{\ell!} f_{n+\ell}(\tau^{\wedge \ell}, x_1, \dots, x_n).$$

Naturality means that the following diagram is commutative:

$$\begin{array}{ccc} H_n(\mathfrak{g}^\tau) & \xrightarrow{B_n^\tau} & \pi_{n+1}(\gamma_\bullet(\mathfrak{g}), \tau) \\ \downarrow H_n(f^\tau) & & \downarrow f_* \\ H_n(\mathfrak{h}^{f_*(\tau)}) & \xrightarrow{B_n^{f_*(\tau)}} & \pi_{n+1}(\gamma_\bullet(\mathfrak{h}), f_*(\tau)) \end{array}$$

We leave the verification to the reader.

5. APPLICATION: RATIONAL MODELS FOR MAPPING SPACES

In this section, we will show that the mapping space $\text{Map}(X, Y_\mathbb{Q})$ is homotopy equivalent to $\gamma_\bullet(A \hat{\otimes} L)$ where A is a cdga model for X and L is a profinite L_∞ -algebra model for Y . Once this is established, Theorem 1.4 will follow by applying Theorem 1.1 and Corollary 1.2 to the L_∞ -algebra $A \hat{\otimes} L$.

The simplicial deRham algebra Ω_\bullet gives rise to a (contravariant) adjunction between simplicial sets and commutative differential graded algebras

$$\text{sSet} \begin{array}{c} \xrightarrow{\Omega} \\ \xleftarrow{\langle - \rangle} \end{array} \text{CDGA}_{\mathbb{Q}}^{\text{op}},$$

$$\Omega(X) = \text{Hom}_{\text{sSet}}(X, \Omega_\bullet), \quad \langle B \rangle = \text{Hom}_{\text{cdga}}(B, \Omega_\bullet).$$

The cdga $\Omega(X)$ is the Sullivan-deRham algebra of polynomial differential forms on X , and $\langle B \rangle$ is the spatial realization of B . It is a fundamental result in rational

homotopy theory that the adjunction induces an equivalence between the homotopy categories of nilpotent rational spaces of finite \mathbb{Q} -type and minimal algebras of finite type, see [4, 34]. The category of cdgas is enriched in simplicial sets via

$$\mathrm{Map}_{cdga}(B, A) = \mathrm{Hom}_{cdga}(B, A \otimes \Omega_\bullet),$$

see [4, §5], and the category of simplicial sets is enriched in itself via

$$\mathrm{Map}_{sSet}(X, Y) = \mathrm{Hom}_{sSet}(X \times \Delta[\bullet], Y).$$

It is natural to ask to what extent the adjunction above is compatible with the simplicial enrichments. This has been answered by Brown and Szczarba.

Theorem 5.1 (Brown-Szczarba [7, Theorem 2.20]). *Let X be a connected simplicial set and let B be a finite type Sullivan algebra. There is a natural homotopy equivalence of Kan complexes*

$$BS: \mathrm{Map}_{cdga}(B, \Omega(X)) \xrightarrow{\sim} \mathrm{Map}_{sSet}(X, \langle B \rangle).$$

Furthermore, the functor $\mathrm{Map}_{cdga}(B, -)$ takes quasi-isomorphisms between commutative differential graded algebras to homotopy equivalences between Kan complexes.

We give a short proof using nerves of L_∞ -algebras below. The following is a consequence.

Theorem 5.2. *Let X be a connected simplicial set and let Y be a nilpotent space of finite \mathbb{Q} -type. If L is a profinite L_∞ -model for Y and A is a cdga model for X , then there is a homotopy equivalence of Kan complexes*

$$\mathrm{Map}(X, Y_\mathbb{Q}) \simeq \gamma_\bullet(A \hat{\otimes} L).$$

Proof. By the Sullivan-deRham localization theorem [4, §11.2] we may take the spatial realization $Y_\mathbb{Q} = \langle C^*(L) \rangle$ as a \mathbb{Q} -localization of Y . The Brown-Szczarba theorem applied to $B = C^*(L)$ yields a homotopy equivalence between $\mathrm{Map}(X, Y_\mathbb{Q})$ and $\mathrm{Map}_{cdga}(C^*(L), A)$. It follows from Lemma 2.3 that there is an isomorphism of simplicial sets $\mathrm{Map}_{cdga}(C^*(L), A) \cong \mathrm{MC}_\bullet(A \hat{\otimes} L)$. Finally, the inclusion $\gamma_\bullet(A \hat{\otimes} L) \rightarrow \mathrm{MC}_\bullet(A \hat{\otimes} L)$ is a homotopy equivalence. \square

Remark 5.3. If A is a finite type cdga and L is a profinite L_∞ -algebra, then there is an isomorphism of complete L_∞ -algebras $A \hat{\otimes} L \cong \mathrm{Hom}(C, L)$ where $C = A^\vee$ is the dual cdg. If either A or L is finite, then $A \hat{\otimes} L \cong A \otimes L$.

Proof of Theorem 5.1. We now embark on a proof of Theorem 5.1. The theorem is reformulated in the language of nerves of L_∞ -algebras as Theorem 5.5 below.

Proposition 5.4. *Let X be a simplicial set and L a complete L_∞ -algebra. There is a natural map*

$$\mu: \mathrm{MC}(\Omega(X) \hat{\otimes} L) \rightarrow \mathrm{Hom}_{sSet}(X, \mathrm{MC}_\bullet(L))$$

which is an isomorphism if X is finite or if L is profinite.

Proof. We define the map for nilpotent L first. Given $\tau \in \mathrm{MC}(\Omega(X) \otimes L)$, define a simplicial map $f: X \rightarrow \mathrm{MC}_\bullet(L)$ as follows. Given an n -simplex $x: \Delta[n] \rightarrow X$, we let $f(x) \in \mathrm{MC}_n(L)$ be the image of τ under the map

$$x^*: \mathrm{MC}(\Omega(X) \otimes L) \rightarrow \mathrm{MC}(\Omega_n \otimes L).$$

It is straightforward to check that f is a simplicial map, and we set $\mu(\tau) = f$. The map μ is evidently an isomorphism for $X = \Delta[n]$. Since $- \otimes L$ commutes with finite limits, the functor $\mathrm{MC}(\Omega(-) \otimes L)$ takes finite colimits to limits. The functor $\mathrm{Hom}_{sSet}(-, \mathrm{MC}_\bullet(L))$ preserves all limits, so it follows that μ is an isomorphism for finite X . On the other hand, if L is finite then $- \otimes L$ commutes with all limits and

in this case we get an isomorphism for arbitrary X . Finally, for complete L the map μ is defined as the map induced on inverse limits

$$\varprojlim \mathbf{MC}(\Omega(X) \otimes L/F_r L) \rightarrow \varprojlim \mathrm{Hom}_{sSet}(X, \mathbf{MC}_\bullet(L/F_r L)),$$

and by the above this is an isomorphism if X is finite or if each $L/F_r L$ is finite. \square

Theorem 5.5. *Let X be a simplicial set and L a non-negatively graded complete L_∞ -algebra. There is a natural homotopy equivalence of Kan complexes*

$$\varphi: \mathbf{MC}_\bullet(\Omega(X) \hat{\otimes} L) \rightarrow \mathrm{Map}(X, \mathbf{MC}_\bullet(L)).$$

Furthermore, the functor

$$\mathbf{MC}_\bullet(-\hat{\otimes} L): \mathrm{CDGA} \rightarrow \mathrm{sSet}$$

takes quasi-isomorphisms to homotopy equivalences between Kan complexes.

Remark 5.6. There is a commutative diagram

$$\begin{array}{ccc} \mathbf{MC}_\bullet(\Omega(X) \hat{\otimes} L) & \xrightarrow{\varphi} & \mathrm{Map}(X, \mathbf{MC}_\bullet(L)) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Map}_{cdga}(C^*(L), \Omega(X)) & \xrightarrow{BS} & \mathrm{Map}(X, \langle C^*(L) \rangle). \end{array}$$

Therefore, in view of Proposition 2.2, Theorem 5.5 is a reformulation of the Brown-Szcharba theorem.

Proof. We define the map φ for nilpotent L first. On n -simplices, the map φ_n is defined as the composite

$$\begin{aligned} \mathbf{MC}_n(\Omega(X) \otimes L) &= \mathbf{MC}(\Omega(\Delta[n]) \otimes \Omega(X) \otimes L) \\ &\xrightarrow{\pi} \mathbf{MC}(\Omega(\Delta[n] \times X) \otimes L) \\ &\xrightarrow{\mu} \mathrm{Hom}_{sSet}(\Delta[n] \times X, \mathbf{MC}_\bullet(L)) \\ &= \mathrm{Map}(X, \mathbf{MC}_\bullet(L))_n, \end{aligned}$$

where π is induced by the natural morphism of cdgas $\Omega(\Delta[n]) \otimes \Omega(X) \rightarrow \Omega(\Delta[n] \times X)$, and μ is the map described in Proposition 5.4. By naturality of all maps involved, φ respects the simplicial structure. For complete L , we define φ to be the induced map on inverse limits

$$\varprojlim \mathbf{MC}_\bullet(\Omega(X) \otimes L/F_r L) \rightarrow \varprojlim \mathrm{Map}(X, \mathbf{MC}_\bullet(L/F_r L)).$$

The proof that φ is a weak equivalence is by reduction to the abelian case. For abelian L , the map φ is a weak equivalence because it follows from Proposition 3.3 that φ is equivalent to the natural map

$$\Gamma_\bullet(\Omega(X) \otimes L[1]) \rightarrow \mathrm{Map}(X, \Gamma_\bullet(L[1])).$$

But $\Gamma_\bullet(L[1])$ is a product of rational Eilenberg-MacLane spaces, and by standard arguments the homotopy groups of the right hand side are given by $H^*(X; \mathbb{Q}) \otimes H_*(L)[1]$. Next, we note that both source and target of φ take central extensions of L_∞ -algebras to principal fibrations, so we can induct on a composition series for L to conclude that φ is a weak equivalence for all nilpotent L_∞ -algebras L . For complete L , we get a levelwise weak equivalence of towers of fibrations

$$\mathbf{MC}_\bullet(\Omega(X) \otimes L/F_r L) \xrightarrow{\sim} \mathrm{Map}(X, \mathbf{MC}_\bullet(L/F_r L)).$$

Hence, we get a weak equivalence when passing to inverse limits. This finishes the proof of the first statement. The proof that $\mathbf{MC}_\bullet(-\hat{\otimes} L)$ takes quasi-isomorphisms to weak equivalences is proved by a similar reduction to the abelian case. \square

6. EXAMPLES

Finite rational cohomology or homotopy. If $H^*(X; \mathbb{Q})$ or $\pi_*(Y) \otimes \mathbb{Q}$ is finite, then for a fixed map $g: X \rightarrow Y$, composition with the \mathbb{Q} -localization map $r: Y \rightarrow Y_{\mathbb{Q}}$ induces a rational homotopy equivalence

$$\text{Map}(X, Y; g) \xrightarrow{\sim_{\mathbb{Q}}} \text{Map}(X, Y_{\mathbb{Q}}; rg).$$

If X has finite dimensional rational cohomology, we can find a finite dimensional cdga model A (see e.g. [14, Example 6, p.146]). Likewise, if $\pi_*(Y) \otimes \mathbb{Q}$ is finite, then we can find a finite dimensional nilpotent L_{∞} -algebra model L for Y . In either of these situations, $A \hat{\otimes} L = A \otimes L$, and given a Maurer-Cartan element $\tau \in \text{MC}(A \otimes L)$ that represents the map $rg: X \rightarrow Y_{\mathbb{Q}}$, the truncated and twisted L_{∞} -algebra $A \otimes L_{\geq 0}^{\tau}$ is an L_{∞} -algebra model for the component $\text{Map}(X, Y; g)$. Equivalently, the Chevalley-Eilenberg construction $C^*(A \otimes L_{\geq 0}^{\tau})$ is a (not necessarily minimal) Sullivan model for $\text{Map}(X, Y; g)$. In particular, we have an isomorphism of graded Lie algebras

$$\pi_{*+1}(\text{Map}(X, Y), f) \otimes \mathbb{Q} \cong H_*(A \otimes L^{\tau}), \quad * \geq 0.$$

Here, $\pi_1 \otimes \mathbb{Q}$ is interpreted as the Malcev completion of the nilpotent group π_1 [31, §A3], and $H_0(A \otimes L^{\tau})$ is given the Campbell-Hausdorff group structure.

Maps into Eilenberg-MacLane spaces. A simply connected space Y is rationally homotopy equivalent to a product of Eilenberg-MacLane spaces if and only if $L = \pi_*(\Omega Y) \otimes \mathbb{Q}$, viewed as an L_{∞} -algebra with zero differential and zero brackets, is an L_{∞} -model for Y . In this case $A \otimes L$ is abelian for any cdga A , and $A \otimes L^{\tau} = A \otimes L$ for any τ . By choosing a quasi-isomorphism of chain complexes $A \rightarrow H^*(A)$ (not necessarily a cdga morphism) we get a quasi-isomorphism of abelian L_{∞} -algebras $A \otimes L \rightarrow H^*(A) \otimes L$. Therefore, all components of $\text{Map}(X, Y_{\mathbb{Q}})$ are rationally homotopy equivalent and they have a minimal abelian L_{∞} -algebra model $(H^*(X; \mathbb{Q}) \otimes \pi_*(\Omega Y))_{\geq 0}$. So each component is a product of rational Eilenberg-MacLane spaces whose homotopy groups are given by

$$\pi_{k+1}(\text{Map}(X, Y_{\mathbb{Q}}), f) \cong (H^*(X; \mathbb{Q}) \otimes \pi_*(\Omega Y))_k, \quad k \geq 0.$$

Inclusions of complex projective spaces. Consider the standard inclusion $i: \mathbb{CP}^n \rightarrow \mathbb{CP}^m$ where $m \geq n \geq 1$. As a cdga model for \mathbb{CP}^n we may choose the cohomology

$$A = H^*(\mathbb{CP}^n; \mathbb{Q}) = \mathbb{Q}[x]/(x^{n+1}), \quad |x| = 2.$$

A minimal L_{∞} -algebra model for \mathbb{CP}^m is given by

$$L = \pi_*(\Omega \mathbb{CP}^m) \otimes \mathbb{Q} = \langle \alpha, \beta \rangle, \quad |\alpha| = 1, \quad |\beta| = 2m,$$

where the only non-vanishing bracket is

$$\frac{1}{(m+1)!} [\alpha^{\wedge m+1}] = \beta.$$

The twisting cochain $\tau = x \otimes \alpha \in A \otimes L$ represents the inclusion $i: \mathbb{CP}^n \rightarrow \mathbb{CP}^m$. Thus, an L_{∞} -model for the component $\text{Map}(\mathbb{CP}^n, \mathbb{CP}^m; i)$ is given by the finite dimensional L_{∞} -algebra $\mathfrak{g} = A \otimes L_{\geq 0}^{\tau}$, and the Chevalley-Eilenberg construction $C^*(\mathfrak{g})$ is a Sullivan model. A basis for \mathfrak{g} is given by

$$\begin{array}{cccccc} 2m & 2m-2 & \cdots & 2m-2n & 1 \\ 1 \otimes \beta & x \otimes \beta & \cdots & x^n \otimes \beta & 1 \otimes \alpha \end{array}$$

The L_{∞} -algebra structure is described by

$$\frac{1}{r!} [(1 \otimes \alpha)^{\wedge r}]_{\tau} = \binom{m+1}{r} x^{m+1-r} \otimes \beta.$$

Since β is central in L , it follows that the elements $x^r \otimes \beta$ are central in \mathfrak{g} , so the only possible non-zero brackets are described by the above. Note that we get zero if $r \leq m - n$ since $x^{n+1} = 0$. In particular, the differential is zero if $m > n$, and in this case the Chevalley-Eilenberg construction $C^*(\mathfrak{g})$ is a *minimal* Sullivan model for $\text{Map}(\mathbb{CP}^n, \mathbb{CP}^m; i)$. It has the following description:

$$\Lambda(z, w_{m-n}, w_{m-n+1}, \dots, w_m), \quad dz = 0, \quad dw_r = z^{r+1}, \quad |z| = 2, \quad |w_r| = 2r + 1.$$

On the other hand, if $m = n$, then we have a non-zero differential $[1 \otimes \alpha]_\tau = (n+1)x^n \otimes \beta$, and hence $\pi_*(\text{Map}(\mathbb{CP}^n, \mathbb{CP}^n); 1) \otimes \mathbb{Q}$ is concentrated in odd degrees (as predicted by Halperin's conjecture, see below) with precisely one basis element each in the degrees $3, 5, \dots, 2n + 1$. The minimal Sullivan model for $\text{Map}(\mathbb{CP}^n, \mathbb{CP}^n; 1)$ is therefore an exterior algebra on the dual of the rational homotopy with zero differential:

$$\Lambda(x_3, x_5, \dots, x_{2n+1}), \quad dx_i = 0, \quad |x_i| = i.$$

On the Halperin conjecture. Let X be an F_0 -space, i.e., a simply connected space with evenly graded rational cohomology such that both $\dim_{\mathbb{Q}} H^*(X; \mathbb{Q}) < \infty$ and $\dim_{\mathbb{Q}} \pi_*(X) \otimes \mathbb{Q} < \infty$. Then X is formal and the cohomology algebra admits a presentation

$$H^*(X; \mathbb{Q}) = \mathbb{Q}[x_1, \dots, x_n] / (f_1, \dots, f_n),$$

where x_1, \dots, x_n are evenly graded generators and f_1, \dots, f_n is a regular sequence [21]. *Halperin's conjecture* says that for such spaces X , the component $\text{aut}_1(X)$ of the space of homotopy self-equivalences of X that contains the identity map is rationally homotopy equivalent to a product of odd dimensional spheres. Equivalently, the rational homotopy groups $\pi_*(\text{aut } X, 1) \otimes \mathbb{Q}$ are concentrated in odd degrees. Using the model in Theorem 1.4, we get a new proof of the following.

Theorem 6.1. (Meier [29]) *Let X be an F_0 -space. Then the Halperin conjecture holds for X if and only if the cohomology algebra $H^*(X; \mathbb{Q})$ admits no derivations of negative degree.*

Proof. A minimal model for X is given by

$$\mathbb{Q}[x_1, \dots, x_n] \otimes \Lambda(y_1, \dots, y_n), \quad dx_i = 0, \quad dy_i = f_i.$$

Here y_i is a generator of odd degree $|y_i| = |f_i| - 1$. The dual L_∞ -algebra L is given by

$$L = L_{\text{odd}} \oplus L_{\text{even}} = \langle \alpha_1, \dots, \alpha_n \rangle \oplus \langle \beta_1, \dots, \beta_n \rangle,$$

where $|\alpha_i| = |x_i| - 1$ are odd and $|\beta_i| = |y_i| - 1$ even. We have an isomorphism of rational vector spaces for every $n \geq 0$,

$$\pi_{n+1}(\text{aut } X, 1_X) \otimes \mathbb{Q} = H_n(A \otimes L, D^\pi), \quad D^\pi(\xi) = \sum_{\ell \geq 2} \frac{1}{\ell!} [\pi^{\wedge \ell}, \xi],$$

where the Maurer-Cartan element π is given by $\pi = \sum_i x_i \otimes \alpha_i + y_i \otimes \beta_i$. Hence, the Halperin conjecture holds for X if and only if

$$H_n(A \otimes L, D^\pi) = 0, \quad \text{for all odd } n > 0.$$

Since L_{even} is central in L , we have that $D^\pi(A \otimes L_{\text{even}}) = 0$. Thus,

$$H_{\text{odd}}(A \otimes L, D^\pi) = \ker(D^\pi: A \otimes L_{\text{odd}} \rightarrow A \otimes L_{\text{even}}).$$

The differential D^π is A -linear, and a calculation yields

$$D^\pi(1 \otimes \alpha_i) = \sum_{j=1}^n \frac{\partial f_j}{\partial x_i} \otimes \beta_j.$$

Therefore, an element $\sum_i p_i \otimes \alpha_i \in A \otimes L_{\text{odd}}$ belongs to the kernel of D^π if and only if

$$\sum_{i=1}^n p_i \frac{\partial f_j}{\partial x_i} = 0, \quad 1 \leq j \leq n.$$

But this is true if and only if $\sum_i p_i \frac{\partial}{\partial x_i}$ defines a derivation of A . Thus, we obtain an isomorphism

$$\ker(D^\pi : A \otimes L_{\text{odd}} \rightarrow A \otimes L_{\text{even}}) \cong \text{Der} A, \quad \sum_{i=1}^n p_i \otimes \alpha_i \mapsto \sum_{i=1}^n p_i \frac{\partial}{\partial x_i}.$$

This shows that $H_n(A \otimes L, D^\pi) = 0$ for all odd $n > 0$ if and only if A admits no negative (in cohomological grading) derivations. \square

Homotopy automorphisms of formal and coformal spaces. In [1], the following characterization of spaces that are both formal and coformal was established.

Theorem 6.2 ([1]). *The following are equivalent for a connected nilpotent space X of finite \mathbb{Q} -type:*

- (1) X is both formal and coformal.
- (2) X is formal and $H^*(X; \mathbb{Q})$ is a Koszul algebra.
- (3) X is coformal and $\pi_*(\Omega X) \otimes \mathbb{Q}$ is a Koszul Lie algebra.

In this situation, homotopy is Koszul dual to cohomology in the sense that

$$\pi_*(\Omega X) \otimes \mathbb{Q} \cong H^*(X; \mathbb{Q})^{!_{\text{Lie}}}.$$

That the cohomology $H^*(X; \mathbb{Q})$ is a Koszul algebra means that it is generated by elements x_i modulo quadratic relations $\sum_i c_{ij} x_i x_j = 0$ such that $\text{Tor}_{s,t}^A(\mathbb{Q}, \mathbb{Q}) = 0$ for $s \neq t$, where the extra grading on Tor is induced by wordlength in the generators x_i . That $\pi_*(\Omega X) \otimes \mathbb{Q} \cong H^*(X; \mathbb{Q})^{!_{\text{Lie}}}$ means that, as a graded Lie algebra, $\pi_*(\Omega X) \otimes \mathbb{Q}$ is generated by classes α_i dual to x_i modulo the *orthogonal relations*: a relation

$$\sum_{i,j} \lambda_{ij} [\alpha_i, \alpha_j] = 0$$

holds if and only if

$$\sum_{i,j} (-1)^{|x_i||\alpha_j|} c_{ij} \lambda_{ij} = 0$$

whenever the coefficients c_{ij} represent a relation among the generators x_i for $H^*(X; \mathbb{Q})$.

The component of the mapping space $\text{Map}(X, X)$ that contains a fixed homotopy self-equivalence is equal to the same component of $\text{aut}(X)$, since any map homotopic to a homotopy equivalence is itself a homotopy equivalence. Moreover, $\pi_1(\text{aut}(X), 1_X)$ is an abelian group as $\text{aut}(X)$ is a monoid. By combining Theorem 1.4 and Theorem 6.2, we obtain the following theorem.

Theorem 6.3. *Let X be formal and coformal nilpotent space such that either $\dim_{\mathbb{Q}} H^*(X; \mathbb{Q}) < \infty$ or $\dim_{\mathbb{Q}} \pi_*(X) \otimes \mathbb{Q} < \infty$. Then there is a finite basis x_1, \dots, x_n for the indecomposables of the cohomology algebra $H^*(X; \mathbb{Q})$ and a dual basis $\alpha_1, \dots, \alpha_n$ for the indecomposables of the homotopy Lie algebra $\pi_*(\Omega X) \otimes \mathbb{Q}$. Setting $\kappa = x_1 \otimes \alpha_1 + \dots + x_n \otimes \alpha_n \in H^*(X; \mathbb{Q}) \otimes \pi_*(\Omega X)$, the derivation $[\kappa, -]$ is a differential, and there are isomorphisms*

$$\pi_{k+1}(\text{aut}(X), 1_X) \otimes \mathbb{Q} \cong H_k(H^*(X; \mathbb{Q}) \otimes \pi_*(\Omega X), [\kappa, -]), \quad k \geq 0.$$

In [2], we use this result to calculate the rational homotopy groups of the space of self-equivalences of highly connected manifolds.

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